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Superintegrable systems with third-order integrals of motion

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Abstract
Two-dimensional superintegrable systems with one third-order and one lower order integral of motion are reviewed. The fact that Hamiltonian systems with higher order integrals of motion are not the same in classical and quantum mechanics is stressed. New results on the use of classical and quantum third-order integrals are presented in sections 5 and 6.

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1. Introduction

The purpose of this paper is to present some new results and insights on superintegrable systems in classical and quantum mechanics, involving integrals of motion that are cubic in the momenta. We also present a review of this field with emphasis on the differences between 'quadratic' and 'cubic' integrability and superintegrability.

We recall that in classical mechanics a Hamiltonian system with Hamiltonian H and integrals of motion X_a,

$$H = \tfrac{1}{2}g_{ik}p_i p_k + V(\vec{x}, \vec{p}), \qquad X_a = f_a(\vec{x}, \vec{p}), \qquad a = 1, \ldots, n-1, \tag{1.1}$$

is called completely integrable (or Liouville integrable) if it allows n integrals of motion (including the Hamiltonian) that are well-defined functions on phase space, are in involution $\{H, X_a\}_p = 0$, $\{X_a, X_b\}_p = 0$, $a, b = 1, \ldots, n-1$ and are functionally independent ($\{,\}_p$ is a Poisson bracket). A system is superintegrable if it is integrable and allows further integrals of motion $Y_b(\vec{x}, \vec{p})$, $\{H, Y_b\}_p = 0$, $b = n, n+1, \ldots, n+k$ that are also well-defined functions on phase space and the integrals $\{H, X_1, \ldots, X_{n-1}, Y_n, \ldots, Y_{n+k}\}$ are functionally independent. A system is maximally superintegrable if the set contains $2n-1$ functions, minimally superintegrable if it contains $n+1$ such integrals. The integrals Y_b are not required to be in evolution with $X_1, \ldots, X_{n-1}$, nor with each other.

The same definitions apply in quantum mechanics but $\{H, X_a, Y_b\}$ are well-defined quantum-mechanical operators, assumed to form an algebraically independent set.

The best known examples of (maximally) superintegrable systems are the Kepler–Coulomb [1, 2] system $V(\vec{x}) = \frac{\alpha}{r}$ and the harmonic oscillator $V(\vec{x}) = \alpha r^2$ [3, 4]. It follows from Bertrand's theorem [5, 6] that these are the only two spherically symmetric examples in Euclidean space.

In both cases the integrals of motion are first- or second-order polynomials in the momenta. The Hamiltonians and integrals are the same in classical and quantum mechanics (after a possible symmetrization).

A systematic search for superintegrable systems in two-dimensional Euclidean space E_2 was started some time ago [7, 8]. Rotational symmetry was not imposed. The integrals were assumed to be second-order polynomials in the momenta. Thus, the Ansatz (in quantum mechanics) was

$$\begin{aligned} H &= \frac{1}{2}(\vec{p})^2 + V(\vec{x}), \qquad \vec{x} = (x_1, x_2), \\ X_a &= \sum_{i,k=1}^{2} \left\{ f_a^{ik}(\vec{x}), p_i p_k \right\} + \sum_{i=1}^{2} g_a^i(\vec{x}) p_i + \phi_a(\vec{x}), \end{aligned} \tag{1.2}$$

where { , } is an anticommutator and we put

$$p_j = -\mathrm{i}\hbar \frac{\partial}{\partial x_j}, \qquad L_3 = x_2 p_1 - x_1 p_2. \tag{1.3}$$

The commutativity requirement $[H, X_a] = 0$ implies that X_a must have the following form:

$$X = aL_3^2 + b(L_3 p_1 + p_1 L_3) + c(L_3 p_2 + p_2 L_3) + d\left(p_1^2 - p_2^2\right) + 2 f p_1 p_2 + \phi(x, y), \tag{1.4}$$

where $a, \ldots, f$ are constants (or X can be a first-order operator).

Using transformations from the Euclidean group E(2) (they leave the form of the Hamiltonian (1.2) invariant) we can transform integral (1.4) into one of the four standard forms. If one such operator X exists then the potential $V(x_1, x_2)$ will allow the separation of variables in Cartesian, polar, parabolic or elliptic coordinates, respectively. If two operators $\{X_1, X_2\}$ commuting with H exist, we obtain four families of superintegrable potentials [7, 8], namely,

$$\begin{aligned} &V_I = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}, \qquad V_{II} = \alpha(x^2 + 4y^2) + \frac{\beta}{x^2} + \gamma y, \\ &V_{III} = \frac{\alpha}{r} + \frac{1}{r^2}\left(\frac{\alpha}{1 + \cos(\phi)} + \frac{\beta}{1 - \cos(\phi)}\right), \; V_{IV} = \frac{\alpha}{r} + \frac{1}{\sqrt{r}}\left(\beta \cos\left(\frac{\phi}{2}\right) + \gamma \sin\left(\frac{\phi}{2}\right)\right). \end{aligned} \tag{1.5}$$

Each of these potentials is multiseparable, i.e. allows the separation of variables in the Schrödinger equation (and in the Hamilton–Jacobi equation) in at least two systems of coordinates.

A similar search for superintegrable systems in three-dimensional Euclidean space E_3 gave a complete classification of all 'quadratically superintegrable' systems in E_3 [9, 10].

Since then many results have been obtained on superintegrable systems in E_n for n arbitrary and for two-dimensional spaces of constant curvature, so-called Darboux spaces (two-dimensional spaces with non-constant curvature allowing at least two Killing tensors) for complex spaces etc [11–15].

The overall picture that emerges is that superintegrable systems with quadratic (or linear) integrals of motion have the following properties that make them interesting from the point of view of physics and mathematics:

(1) All finite classical trajectories are closed (more generally, they are constrained to an $n-k$ dimensional manifold in phase space [16]).
(2) They are all multiseparable (in classical and quantum mechanics).
(3) Their quantum energy levels are degenerate.
(4) The Schrödinger equation for quadratically superintegrable systems in E_2 is exactly solvable and it has been conjectured that this is true in general [17]. The conjecture is true in all cases so far studied.
(5) The same potentials are superintegrable in classical and quantum mechanics.
(6) The integrals of motion form non-Abelian algebras under commutation, or Poisson commutation, respectively. These algebras may be finite-dimensional, or infinite-dimensional Lie algebras. The infinite-dimensional ones have a special structure and it is often advantageous to view them as polynomial algebras, more specifically quadratic ones [18–20].

It would be interesting to establish which of the above properties hold for all superintegrable systems and which are restricted to Hamiltonians of the form (1.1) with quadratic integrals of motion. Other systems that have been studied include velocity-dependent potentials [21–25] and particles with spin [26]. In this paper we review the case of a scalar potential as in (1.1) but with third-order integrals of motion. We also present some new results.

2. Integrability with third-order integral of motion

In 1935 J Drach published two articles on two-dimensional Hamiltonian systems with third-order integrals of motion [27]. The main features of these papers are:

(1) The author found 10 such integrable potentials, each depending on 3 constants, but not involving any arbitrary functions.
(2) The results were obtained in classical mechanics and are in general not valid in quantum mechanics.
(3) The space considered was a complex Euclidean space $E_2(\mathbb{C})$.
(4) The articles are very short and 'condensed'. Basically they are research announcements, but no detailed follow-up articles were published.
(5) Some assumptions were made in the calculation so it is not clear whether the obtained list is complete.

More recently, it was shown by Rañada [28] and Tsiganov [29] that 7 of the 10 Drach potentials are actually 'reducible'. The 7 corresponding Hamiltonian systems are superintegrable, in that they allow two second-order integrals. Their Poisson commutator is the third-order integral found by Drach.

Since then many examples of systems with third- and higher order integrals have been published, mainly in classical mechanics [30–34].

In 1998, J Hietarinta published a remarkable article 'Pure quantum integrability' [35] in which he showed that potentials exist that are integrable in quantum mechanics but have free motion as their classical limit. The integrals of motion are third- or higher order polynomials in the momenta and the potentials have the form $V(x, y) = \hbar^2 f(x, y)$, where $\hbar$ is the Planck constant and $f(x, y)$ remains finite in the limit $\hbar \to 0$.

A systematic search for integrable and superintegrable systems with third order integrals of motion in both classical and quantum mechanics was initiated in [36]. The problem was formulated in quantum mechanics; the classical case was considered as a limit for $\hbar \to 0$. The Hamiltonian and first integral have the form

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \tag{2.1}$$

$$X = \frac{1}{2}\sum_{j,k} f_{jk}(x_1, x_2) p_1^j p_2^k \qquad 0 \leqslant j + k \leqslant 3, \tag{2.2}$$

where p_j are as in (1.3) and the potential V and functions f_{jk} are to be determined. The commutativity requirement $[H,X] = 0$ implies that for any potential the third-order integral X will have the form

$$X = \sum_{i+j+k=3} A_{ijk}\left\{L_3^i, p_1^j p_2^k\right\} + \{g_1(x_1, x_2), p_1\} + \{g_2(x_1, x_2), p_2\}, \tag{2.3}$$

where $\{,\}$ is an anticommutator, L_3 is the angular momentum, as in equation (1.3). The constants A_{ijk} and functions V, g_1 and g_2 are subject to further determining equations:

$$\begin{aligned} &(g_1)_{x_1} = 3f_1(x_2)V_{x_1} + f_2(x_1, x_2)V_{x_2}, \qquad (g_2)_{x_2} = f_3(x_1, x_2)V_{x_1} + 3f_4(x_1)V_{x_2}, \\ &(g_1)_{x_2} + (g_2)_{x_1} = 2(f_2(x_1, x_2)V_{x_1} + f_3(x_1, x_2)V_{x_2}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} g_1 V_{x_1} + g_2 V_{x_2} = {} & \frac{\hbar^2}{4}(f_1 V_{x_1x_1x_1} + f_2 V_{x_1x_1x_2} + f_3 V_{x_1x_2x_2} + f_4 V_{x_2x_2x_2}) \\ & + 8A_{300}(x_1 V_{x_2} - x_2 V_{x_1}) + 2(A_{210} V_{x_1} + A_{201} V_{x_2}). \end{aligned} \tag{2.5}$$

The functions f_i are defined in terms of the constants A_{ijk} as

$$\begin{aligned} f_1 &= -A_{300}x_2^3 + A_{210}x_2^2 - A_{120}x_2 + A_{030}, \\ f_2 &= 3A_{300}x_1x_2^2 - 2A_{210}x_1x_2 + A_{201}x_2^2 + A_{120}x_1 - A_{111}x_2 + A_{021}, \\ f_3 &= -3A_{300}x_1^2x_2 + A_{210}x_1^2 - 2A_{201}x_1x_2 + A_{111}x_1 - A_{102}x_2 + A_{012}, \\ f_4 &= A_{300}x_1^3 + A_{210}x_1^2 + A_{102}x_1 + A_{003}. \end{aligned} \tag{2.6}$$

We can sum up the results on third-order integrability as follows.

(1) The leading part of the integral X lies in the enveloping algebra of the Euclidean Lie algebra $e(2)$.
(2) Even and odd terms in X commute with H separately. We require the existence of a third-order integral. A second-order one of the form (1.4) may, or may not exist.
(3) The 10 constants and 3 functions g_1, g_2 and V are to be determined from the (overdetermined) systems of 4 equations (2.4) and (2.5). The fact that the system is overdetermined implies that $V(x, y)$ will satisfy some compatibility conditions. Indeed, one can obtain 1 linear third-order equation for V and 3 nonlinear ones [36]. The existence of a third-order integral is hence much more constraining than the existence of a second-order one (in agreement with Drach's results).
(4) The classical and quantum cases differ. Indeed, the determining equation (2.4) is the same in both cases, but equation (2.5) involves the Planck constant $\hbar$. In the classical limit $\hbar \to 0$ it simplifies greatly (the right-hand side goes to zero).

Instead of attempting to find a general solution we turn to a simpler problem, namely that of superintegrability. We shall consider potentials $V(x, y)$ for which two integrals of motion exist, one third-order one and one that is either a first, or a second-order polynomial in the momenta.

3. Systems with one first-order and one third-order integral

Let us now assume that Hamiltonian (2.1) allows a first-order integral of motion Y. This means that the potential has a geometric symmetry: it is invariant either under rotations, or translations in one direction. With no loss of generality we can consider just two representative cases:

$$\begin{aligned} &(1)\quad V = V(r), \qquad Y = L_3, \qquad r = \sqrt{x^2+y^2}, \\ &(2)\quad V = V(x), \qquad Y = P_2. \end{aligned} \tag{3.1}$$

Equations (2.4) and (2.5) for the existence of a third-order integral X greatly simplify in both cases.

Let us first consider classical mechanics, i.e. $\hbar = 0$ in equation (2.5). Solving systems (2.4) and (2.5) for the above potentials, we obtain

$$V_1 = \frac{\alpha}{r}, \qquad V_2 = \alpha r^2, \qquad V_3 = ax, \qquad V_4 = \frac{a}{x^2}. \tag{3.2}$$

All of these potentials are well known [1–8, 37] as quadratically superintegrable. Indeed, the third-order integral in all these cases is the Poisson commutator of a second-order integral with the square of the first-order one.

In quantum mechanics we again obtain the potentials (3.2) but in addition we obtain a potential $V(x)$ satisfying

$$\hbar^2(V'(x))^2 = 4V^3 + \alpha V^2 + \beta V + \gamma \equiv 4(V-A_1)(V-A_2)(V-A_3), \tag{3.3}$$

where α, β and γ (and A_1, A_2, A_3) are constants. Equation (3.3) is solved in terms of elliptic functions. Depending on properties of the roots A_i and on the initial conditions, we obtain 3 types of solutions:

$$\begin{aligned} &V = (\hbar\omega)^2 k^2 sn^2(\omega x, k), \qquad V = \frac{(\hbar\omega)^2}{2(cn(\omega x, k)+1)}, \\ &V = (\hbar\omega)^2 \frac{1}{sn^2(\omega x, k)}, \end{aligned} \tag{3.4}$$

where $sn(\omega x, k)$ and $cn(\omega x, k)$ are Jacobi elliptic functions. If two of the roots A_i coincide, we get potentials expressed in terms of elementary functions:

$$\begin{aligned} &V = \frac{(\hbar\omega)^2}{(\cosh(\omega x))^2}, \qquad V = \frac{(\hbar\omega)^2}{(\sinh(\omega x))^2}, \\ &V = \frac{(\hbar\omega)^2}{\sin^2(\omega x)}. \end{aligned} \tag{3.5}$$

We see explicitly that for $\hbar \to 0$ all these potentials vanish. Potentials (3.4) and (3.5) are ‘irreducible’ and genuinely superintegrable. The two algebraically independent integrals are

$$Y = P_2, \qquad X = \{L_3, p_1^2\} + \{(\sigma - 3V)y, p_1\} + \left\{-\sigma x + 2xV + \int V(x)\,\mathrm{d}x, p_2\right\}, \tag{3.6}$$

$\sigma = A_1 + A_2 + A_3$.

Even though the potentials *V(x)* in (3.4) and (3.5) are one-dimensional (depend on x alone), their superintegrability is a two-dimensional phenomenon, since the integral X involves rotations L_3.

4. Systems with one second and one third-order integral

Let us now assume that Hamiltonian (2.1) allows one second-order integral of motion Y. This integral will have the form (1.4). As mentioned in the introduction, four classes of such Hamiltonians exist [7, 8]. Here we shall restrict to one of the four classes, namely to potentials allowing the separation of variables in Cartesian coordinates. Thus we have

$$Y = \tfrac{1}{2}(p_1^2 - p_2^2) + V_1(x) - V_2(y), \qquad V(x, y) = V_1(x) + V_2(y). \tag{4.1}$$

We substitute the above potential into the determining equations (2.4) and (2.5) in order to determine potentials that also allow a third-order integral. The obtained system of equations is quite manageable and was analyzed by S Gravel [38].

The results in the classical and quantum cases are very different. To illustrate the difference, let first consider an example. One particular solution of the determining system (2.4), (2.5) is the potential

$$V(x, y) = \frac{\omega^2}{2} y^2 + V(x), \tag{4.2}$$

where $V(x)$ satisfies a fourth-order nonlinear ordinary differential equation

$$\hbar^2 V^{(4)} = 12\omega^2 x V' + 6(V^2)'' - 2\omega^2 x^2 V'' + 2\omega^2 x^2. \tag{4.3}$$

We mention that this equation is also obtained as a nonclassical reduction of the Bousinesq equation [39, 40]. For $\hbar \neq 0$ this equation is solved in terms of the Painlevé transcendent [41] $P_{IV}\left(x, -\frac{4\omega^2}{\hbar}\right)$. The classical limit is singular ($\hbar \to 0$ reduces the order of the equation). In the classical case we can reduce to quadratures and obtain a quartic algebraic equation for $V(x)$:

$$\begin{aligned} &-9V^4(x) + 14\omega^2 x^2 V^3(x) + \left(6d - \frac{15}{2}\omega^4 x^4\right) V^2(x) \\ &\quad + \left(\frac{3}{2}\omega^6 x^6 - 2d\omega^2 x^2\right) V(x) + cx^2 - d^2 - d\frac{\omega^2}{2}x^4 - \frac{1}{16}\omega^8 x^8 = 0. \end{aligned} \tag{4.4}$$

For special values of the constants $a, \ldots, d, \omega$ namely

$$c = \frac{8\omega^8 b^3}{36}, \qquad d = \frac{\omega^4 b^2}{3^3} \tag{4.5}$$

equation (4.4) has a double root and we obtain

$$V_{1,2} = \frac{\omega^2}{18}(2b + 5x^2 \pm 4x\sqrt{b + x^2}), \qquad V_3 = V_4 = \frac{\omega^2}{2}x^2 - \frac{\omega^2 b}{3^3}. \tag{4.6}$$

A complete analysis of equations (2.4) and (2.5) is given in [38]. All together 8 classical superintegrable systems that are separable in cartesian coordinates and allow at least one third-order integral of motion exist. Three of them are reducible (the harmonic oscillator and potentials V_I and V_{II} of equation (1.5)). The five irreducible classical potentials are

$$\begin{aligned} V &= \frac{\omega^2}{2}(9x^2 + y^2), \\ V &= \beta_1^2\sqrt{|x|} + \beta_2^2\sqrt{|y|}, \\ V &= a^2|y| + b^2\sqrt{|x|}, \\ V &= \frac{\omega^2}{2}y^2 + V(x), \qquad V = a|y| + f(x), \end{aligned}$$

where $V(x)$ satisfies equation (4.4) and $f(x)$ satisfies $f^3 - 2bxf^2 + b^2x^4 f - d = 0$. The quantum case is much richer: 21 superintegrable cases of the considered type exist, 13 of them irreducible. The potentials are expressed in terms of rational functions in 6 cases, elliptic functions in 2 cases and Painlevé transcendents [41] P_I, P_{II} and P_{IV} in 5 cases. Let us just present the irreducible potentials.

Rational function potentials:

$$V = \hbar^2\left[\frac{x^2+y^2}{8a^4} + \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2}\right],$$
$$V = \hbar^2\left[\frac{1}{8a^4}(x^2+y^2) + \frac{1}{y^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}\right],$$
$$V = \hbar^2\left[\frac{1}{8a^4}(x^2+y^2) + \frac{1}{(y+a)^2} + \frac{1}{(y-a)^2} + \frac{1}{(x+a)^2} + \frac{1}{(x-a)^2}\right],$$
$$V = \frac{\omega^2}{2}(9x^2+y^2),$$
$$V = \frac{\omega^2}{2}(9x^2+y^2) + \frac{\hbar^2}{y^2},$$
$$V = \hbar^2\left[\frac{9x^2+y^2}{8a^4} + \frac{1}{(y-a)^2} + \frac{1}{(y+a)^2}\right].$$

Elliptic function potentials:

$$V = \hbar^2 P(y) + V(x),\ V(x) \text{ is arbitrary},$$
$$V = \hbar^2(P(y) + P(x)),\ P(x) \text{ is a Weierstrass elliptic function.}$$

Painlevé transcendent potentials:

$$V = \hbar^2\big(\omega_1^2 P_I(\omega_1 x) + \omega_2^2 P_I(\omega_2 y)\big),$$
$$V = ay + \hbar^2\omega_1^2 P_I(\omega_1 x),$$
$$V = bx + ay + (2\hbar b)^{\frac{2}{3}} P_{II}^2\left(\left(\frac{2b}{\hbar^2}\right)^{\frac{1}{3}} x\right),$$
$$V = ay + (2\hbar^2 b^2)^{\frac{1}{3}}\left(P_{II}'\left(\left(\frac{-4b}{\hbar^2}\right)^{\frac{1}{3}} x\right) + P_{II}^2\left(\left(\frac{-4b}{\hbar^2}\right)^{\frac{1}{3}} x\right)\right),$$
$$V = a(x^2+y^2) + \frac{\hbar^2}{2}\left(P_{IV}'\left(\frac{-8a}{\hbar^2}x\right) - \frac{1}{2}\sqrt{8a}\,P_{IV}^2\left(\frac{-8a}{\hbar^2}x\right) - \frac{1}{2}\sqrt{8a}x\,P_{IV}\left(\frac{-8a}{\hbar^2}x\right)\right).$$

The occurence of Painlevé transcendents as superintegrable potentials seems somewhat surprising. It is less so once we remember the relation between the Schrödinger equation and the Korteweg–de Vries equation [42]. Solutions of the KdV include Painlevé transcendents. Painlevé transcendent potentials have already occurred in other contexts [43–45].

5. Trajectories for the classical systems

The trajectories for all eight superintegrable classical potentials were presented in [46]. The trajectories can be obtained by integrating the equations of motion and then eliminating the time variable from the result. Alternatively, the trajectories can be obtained directly from the integrals of motion. Indeed, in all cases we have

$$\tfrac{1}{2}p_1^2 + f(x) = E_1, \qquad \tfrac{1}{2}p_2^2 + g(x) = E_2, \tag{5.1}$$

$$X = \mu p_1^3 + \nu p_1^2 p_2 + \rho p_1 p_2^2 + \sigma p_2^3 + \phi p_1 + \psi p_2 = K, \tag{5.2}$$

where $\mu, \ldots, \sigma$ are known low order polynomials in x and y, $f(x)$, $g(x)$, $\phi(x, y)$ and $\psi(x, y)$ are known functions and E_1, E_2 and K are constants depending on the initial conditions.

From equation (5.1) we express P_1 and P_2 in terms of E_1, E_2, $f(x)$ and $g(y)$ and substitute into equation (5.2). This directly gives us an equation for the trajectory (not necessarily in a convenient form). If the functions $f(x)$ and $g(y)$ satisfy $f(x) \geqslant 0$, $g(y) \geqslant 0$, for $x^2 > x_0$, $y^2 > y_0$ for some constants x_0 and y_0, then the motion will be bounded.

From the figures of [46] we see that the finite trajectories are all closed, as predicted by the general theory.

6. Algebras of the quantum integrals of motion

The integrals of motion in quantum mechanics form associative polynomial (cubic) algebras with respect to Lie commutation. The commutator relations in all cases are of the form

$$[A, B] = C, \qquad [A, C] = \alpha B, \qquad [B, C] = \beta A^3 + \gamma A^2 + \delta A + \epsilon, \tag{6.1}$$

where γ, δ and ϵ are polynomials in H (for the classical cubic algebras of Poisson brackets see [46]).

Inspired by the work of Daskaloyannis *et al* on quadratic algebras [18, 47, 48] we construct a realization of the cubic algebra (6.1) using a deformed oscillator algebra and Fock space basis $\{b^t, b, N\}$. We put

$$[N, b^t] = b^t, \qquad [N, b] = -b, \qquad b^t b = \Phi(N), \qquad bb^t = \Phi(N+1), \tag{6.2}$$

where $\Phi(N)$ is the so-called structure function. To construct representations of the cubic algebra we impose conditions on $\Phi(N)$. We require that a natural number p should exist, such that

$$\Phi(p+1) = 0, \tag{6.3}$$

and further we impose

$$\Phi(0) = 0, \qquad \Phi(x) > 0, \qquad x > 0. \tag{6.4}$$

We put

$$A = A(N), \qquad B = b(N) + b^t \rho(N) + \rho(N) b, \tag{6.5}$$

where the functions $A(N)$, $b(N)$, $\rho(N)$ and $\Phi(N)$ are to be determined from the cubic algebra (6.1).

The next step is to construct a finite-dimensional representation of this algebra with A, N and K (the Casimir operator) diagonal. We obtain a realization of a parafermionic oscillator

$$\begin{aligned}
&N|k, n\rangle = n|k, n\rangle, \qquad K|k, n\rangle = k|k, n\rangle, \\
&A|k, n\rangle = A(k, n)|k, n\rangle, \\
&A|k, n\rangle = \sqrt{\delta}(n+u)|k, n\rangle, \\
&\Phi(0, u, k) = 0, \qquad \Phi(p+1, u, k) = 0.
\end{aligned} \tag{6.6}$$

A complete discussion of this approach to quantum superintegrable systems with cubic integrals of motion will be presented elsewhere.

Here we just present one example showing how one can obtain energy spectra. Let us consider one of the rational quantum potentials of section 4, namely

$$V = \hbar^2 \left[\frac{x^2 + y^2}{8a^4} + \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2} \right]. \tag{6.7}$$

The Casimir operator K and structure function $\Phi(x)$ are in this case found to be

$$K = -16\hbar^2 H^4 + 32\frac{\hbar^4}{a^2}H^3 + 16\frac{\hbar^6}{a^4}H^2 - 40\frac{\hbar^8}{a^6}H - 3\frac{\hbar^{10}}{a^8}, \tag{6.8}$$

$$\Phi(x) = \left(\frac{-\hbar^8}{a^4}\right)\left(x+u-\left(\frac{-a^2E}{\hbar^2}-\frac{1}{2}\right)\right)\left(x+u-\left(\frac{a^2E}{\hbar^2}+\frac{1}{2}\right)\right) \\ \times\left(x+u-\left(\frac{-a^2E}{\hbar^2}+\frac{3}{2}\right)\right)\left(x+u-\left(\frac{-a^2E}{\hbar^2}+\frac{5}{2}\right)\right). \tag{6.9}$$

The condition $\Phi(0, u, k) = 0$ is now an equation for u and $\Phi(p+1, u, k) = 0$ provides the energy spectrum. In the case of the potential (6.7) the constant a can be real, or pure imaginary ($V(x, y)$ is real in both cases). The energy spectrum we obtain is

$$E = \frac{\hbar(p+2)}{2a_0^2} \qquad a = \mathrm{i}a_0, \qquad a_0 \in \mathbb{R}, \qquad p \in \mathbb{N},$$

$$E = \frac{\hbar(p+3)}{2a^2} \qquad a \in \mathbb{R}, \qquad p \in \mathbb{N}.$$

7. Conclusion

The main conclusion that we draw at this stage is that going beyond quadratic integrability for scalar potentials opens new horizons and poses new questions.

One of the questions is: What does one do with integrals of motion that do not lead to the separation of variable in the equations of motion? A partial answer is given in this paper and is very different in classical and quantum mechanics. We have seen that the integrals can be used to calculate trajectories directly without solving the classical equation of motion. In quantum mechanics one uses the algebra of integrals of motion to obtain information about energy spectra and wavefunctions.

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